

AN EMPIRICAL SHEAR STRESS EQUATION FOR GENERAL SOLID–FLUID MIXTURE FLOWS

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Abstract—Shear stress in non-cohesive solid–fluid mixture flow is due to impact among solid particles, viscous interaction between the solid and the fluid and turbulence in the interstitial fluid. Bagnold, based on analogy with the kinetic theory of gas, derived an equation for the shear stress due to the impact of solid particles. It includes a coefficient which should change with different coefficients of restitution of solid materials. However, Bagnold's equation has difficulties in correlating with experiments by other investigators. A new equation for the viscous interaction between solid and fluid is presented here. The new equation and the modified impact coefficient compare well with several sets of experiments by different investigators covering a wide range of physical variables of the mixture flow.

Key Words: two-phase flow, fluid–solid, shear stress, granular flow, sediment transport

1. INTRODUCTION

Solid–fluid mixture flow is usually studied along two distinct modes of transport, i.e. suspension and contact transport (also known as bed load). Research on suspension transport was conducted by Rouse (1937), Einstein & Chien (1955) and Ippen (1971) to name a few. Contact transport was studied by Einstein (1942), Bagnold (1954, 1973), Takahashi (1980), Shibata & Mei (1986) and Julien & Lan (1991), among others. Shen & Ackermann (1982) and Savage & Sayed (1984) derived constitutive equations for solid–fluid mixture flows. These equations are complicated and are not easy to use. Bagnold (1954, 1973) derived an equation for the shear stress due to the impact of solid particles and conducted extensive tests in a Couette apparatus. Bagnold's work is now considered classic in high concentration granular flow research.

Shear stress in solid–fluid flow should include: (1) cohesion among solid particles; (2) impact among solid particles τ_i ; (3) viscous interaction between the solid particles and the surrounding fluid τ_v ; and (4) turbulence stress in the interstitial fluid τ_r . The total shear stress is the sum of these four types of stresses. Cohesive stress is due to electrochemical interaction of molecules or mechanical entanglement among solids in solutions containing fine clay, long-chain polymers, slender rods or fibrillous materials. This paper deals with non-cohesive solid materials.

2. PARTICLE IMPACT SHEAR STRESS τ_i

Bagnold (1954), based on the frequency and the momentum exchange of a sphere during collisions and analogy with the kinetic theory of gas, derived the following equation for the impact shear stress:

$$\tau_i = a_i \rho_s \left(\lambda D \frac{du}{dy} \right)^2, \quad [1]$$

in which a_i is the impact coefficient. Bagnold determined for his mixture $a_i = 0.0128$. The momentum exchange during an impact is a function of the coefficient of restitution for the solid particle. Thus, the impact coefficient will vary with the solid material. ρ_s is the density of the spherical particle, D is the diameter of the sphere, du/dy is the velocity gradient of the mixture and λ is the ratio of sphere diameter D to the mean gap between them S , or $\lambda = D/S$.

Assuming the spheres are dispersed uniformly in the mixture, as shown in figure 1, the distance between the centers of neighboring spheres is $D + S$. Defining a diameter multiplier b , such that $bD = D + S$, then

$$b = 1 + \frac{1}{\lambda} \tag{2}$$

The volumetric concentration of solid C can be expressed in terms of the multiplier b :

$$C = \frac{C_0}{b^3} = \frac{C_0}{\left(1 + \frac{1}{\lambda}\right)^3}, \tag{3}$$

where C_0 is the theoretical maximum concentration of solid spheres when $S = 0$. For uniform spheres, the theoretical maximum concentration (cannon ball piling) in an infinite domain is $C_0 = 0.74$. In the finite volume of the test apparatus, C_0 should be less. Bagnold used $C_0 = 0.72$ on his data from $C = 0.135$ to 0.555 . Since the difference between $C_0 = 0.74$ and 0.72 is not large, $C_0 = 0.72$ is used in this analysis to avoid changing Bagnold's data. Equation [3] can be rewritten as

$$\lambda = \frac{1}{\left(\frac{C_0}{C}\right)^{1/3} - 1} \tag{4}$$

Figure 2 reproduces the result of Bagnold (1954, figure 5) on measured granular shear stress $\tau_v + \tau_i$ vs [1]. The grains were spheres of a nearly 50% mixture of paraffin wax and lead stearate such that their density was equal to water, the interstitial fluid. The spheres had a uniform diameter of 1.32 mm. The sum of impact and viscous stresses ($\tau_v + \tau_i$) was obtained by subtracting the apparatus friction and shear in the fluid τ_f , estimated as the linear porosity $S/(D + S) = 1/(1 + \lambda)$ times the plain-fluid shear, from the total stress measured.

As shown in figure 2, the data points for C from 0.135 to 0.555 ($\lambda = 1.3$ to 11) gather around a single curved line. When the concentration is greater than 0.555 ($\lambda = 11$), the measured shear stress is higher than the line. This increase in stress for $C = 0.623$ ($\lambda = 20$) is attributed to the mechanical jamming or locking of particles at such high concentration in the confined test apparatus.

Impacts among solid grains also generate normal stresses (pressure). Bagnold (1954) derived the impact pressure P_i as

$$P_i = a_p \rho_s \left(\lambda D \frac{du}{dy} \right)^2, \tag{5}$$

where the impact pressure coefficient $a_p = 0.04$. Viscous effects do not generate pressure. According to [5], the impact pressure is about 3.1 times the impact shear stress. For a surge of debris flow

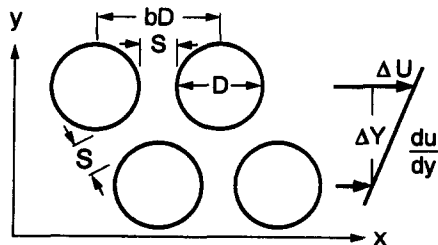


Figure 1. Particle arrangement.

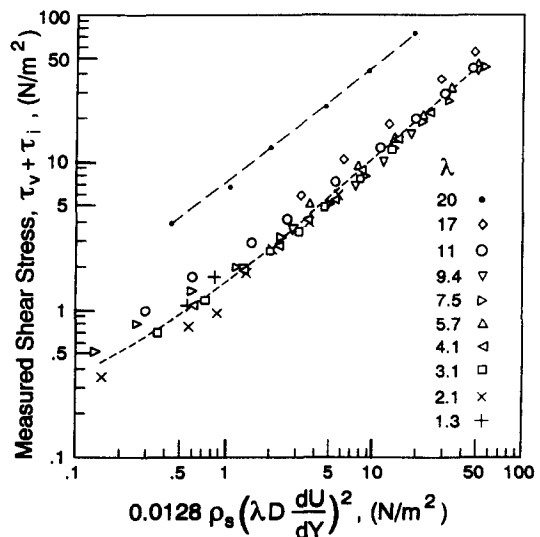


Figure 2. Measured shear stresses vs [1] (Bagnold 1954).

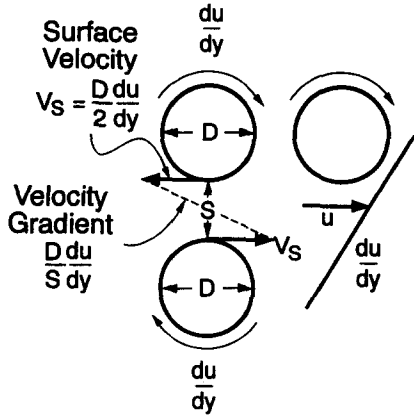


Figure 3. Rotation of spheres and the velocity gradient in the interstitial fluid.

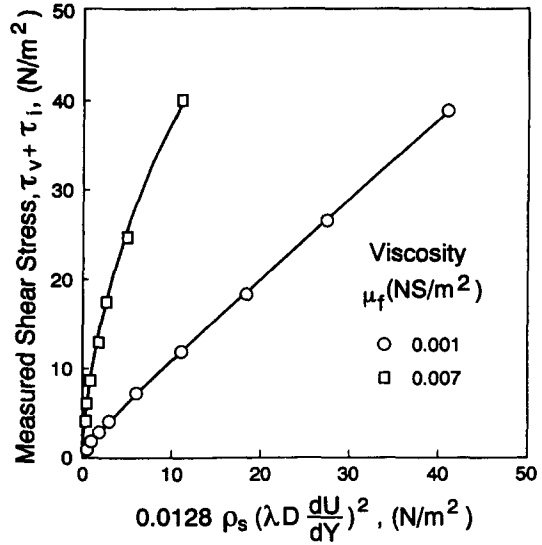


Figure 4. Comparison of measured shear stresses for two different velocities at $C = 0.555$ (Bagnold 1954), with [1].

down a slope, larger particles generate much larger normal stress which pushes them up to the top surface. The faster velocity at the surface then collects the larger particles at the leading edge of the debris flow, as is often observed in the field.

3. VISCOUS INTERACTIVE STRESS ON PARTICLES τ_v

A new equation for the viscous interaction between solid and fluid has been developed. In a mixture flow, solid particles tend to move at the minimum relative velocity with respect to the surrounding fluid, meaning a sphere would have to rotate at the velocity gradient of the mixture, du/dy , or at least in proportion to it. The surface velocity of a rotating sphere is $V_s = (D/2) du/dy$, as shown in figure 3. The surface velocity of neighboring spheres is equal but opposite in direction. The total surface velocity difference across the gap S is $V_s = D du/dy$. Therefore, the average velocity gradient in the interstitial fluid should be proportional to $(du/dy)_f \sim (D/S) du/dy = \lambda du/dy$.

The number of spherical grains per unit area in the horizontal plane is $1/b^2 D^2$. The surface area of each sphere facing the lower layer is $\pi D^2/2$. The surface area of spheres per unit horizontal area of the mixture is $(1/b^2 D^2)(\pi D^2/2) = \pi/(2b^2) = (\pi/2)\lambda^2/(1 + \lambda)^2$ using [2]. The viscous shear stress is proportional to the factor of dynamic viscosity of the fluid μ_f , the average velocity gradient of the interstitial fluid $\lambda du/dy$, and the surface area of spheres per unit horizontal area of the mixture. Therefore,

$$\tau_v \sim \mu_f \frac{\pi}{2} \frac{\lambda^3}{(1 + \lambda)^2} \frac{du}{dy} = a_v \mu_f \frac{\lambda^3}{(1 + \lambda)^2} \frac{du}{dy}, \tag{6}$$

where a_v is a constant called the viscous coefficient. Equation [6] is different from Bagnold's viscous shear equation $\tau_v = 2.25 \mu_f \lambda^{3/2} du/dy$. Combining [6] and [1],

$$\tau_v + \tau_i = a_v \mu_f \frac{\lambda^3}{(1 + \lambda)^2} \frac{du}{dy} + a_i \rho_s \left(\lambda D \frac{du}{dy} \right)^2. \tag{7}$$

The coefficients a_v and a_i can be determined from experimental results. For high solid concentrations, the interstitial fluid can remain laminar even at high velocities. The solids damp turbulence in the mixture flow. When turbulence sets in, the shear stress should be proportional to $(du/dy)^2$, the same as the impact stress.

In Bagnold's (1954) tests at a concentration of 0.555, in addition to water, glycerol and alcohol were added to make the dynamic viscosity 7 times that of water. Figure 4 reproduces the result shown in Bagnold's paper (1954, figure 6). It is clear that viscous shear stress is very important for lower shear rates.

Using the least square method on Bagnold's data at concentration $C = 0.555$ for two widely different viscosities of the fluid, the two coefficients in [7] were determined:

$$\text{viscous coefficient } a_v = 8.1 \tag{8}$$

and

$$\text{impact coefficient } a_i = 0.009. \tag{9}$$

Figure 5 demonstrates the close agreement between the measured shear stresses by Bagnold and [7] for different fluid viscosities. The viscous interaction is a surface phenomenon, there is no reason for the viscous coefficient ($a_v = 8.1$) to change for different solids. It should remain constant while the impact coefficient a_i changes with different coefficients of restitution of solid particle.

Bagnold's data for concentrations from 0.135 to 0.623 is plotted in figure 6. The data points for concentrations of 0.135–0.555 ($\lambda = 1.3$ to 11) form a single straight line which [7] predicts well. When $C > 0.555$, the measured shear stresses are higher than predicted by [7]. This increase of stresses for $C = 0.623$ ($\lambda = 20$) is attributed to the mechanical jamming of particles at such high concentrations in the confined test apparatus, as mentioned before.

The viscous shear stress of [6] is based on the tangential components of the velocity at the nearest points of neighboring spinning spheres. This model predicts a velocity gradient in the interstitial fluid of $(du/dy)_f \sim \lambda du/dy$. One of the reviewers suggests another approach: in a concentrated system, there is a component of relative velocity between the center of neighboring spheres in the y -direction and intense shearing occurs in the squeeze fluid film between two particles. The relative velocity between neighboring centers is $\Delta U = (S + D) du/dy$, figure 3. The gap between two particles is S . The velocity gradient in the gap is $(du/dy)_f \sim \Delta U/S = (1 + \lambda) du/dy$. When multiplied by μ_f and the surface area of spheres per unit area of the mixture $(\pi/2)\lambda^2/(1 + \lambda)^2$, the viscous shear stress is

$$\tau_v = a'_v \mu_f \frac{\lambda^2}{1 + \lambda} \frac{du}{dy}. \tag{6a}$$

Using the least-square method on the same data, the viscous coefficient is determined as $a'_v = 7.4$, while the impact coefficient remains the same at $a_i = 0.009$.

For concentrations from $C = 0.48$ to 0.623 ($\lambda = 7$ to 20), the difference between [6] and [6a] is small, less than 5%. In low concentrations, the particles are known to be spinning. For $C < 0.48$, [6] agrees with the data better than [6a]. Equation [6] will be used in this paper.

Savage & McKeown (1983) and Savage & Sayed (1984) conducted shear tests on different solid materials and fluids. Savage & McKeown (1983) used neutrally buoyant polystyrene beads in salt

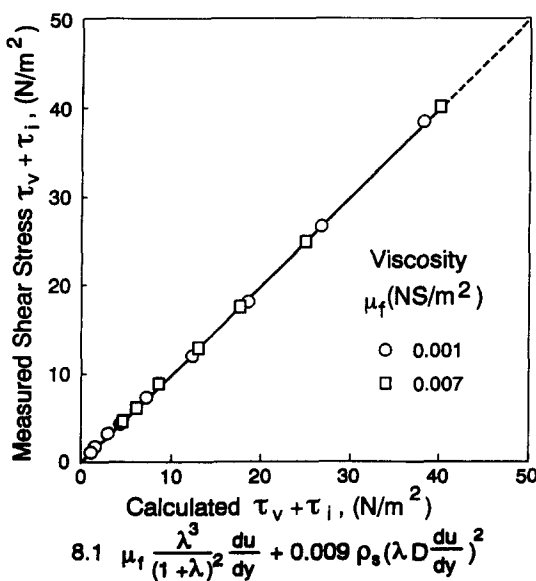


Figure 5. Comparison of measured shear stresses for two different viscosities at $C = 0.555$ (Bagnold 1954) with [7].

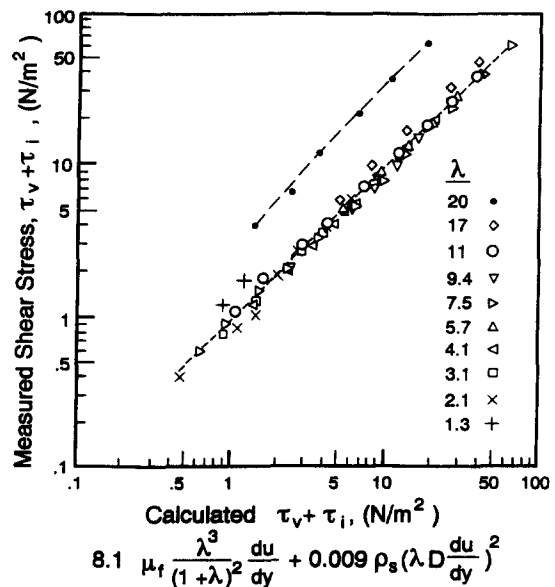


Figure 6. Comparison of measured shear stresses (Bagnold 1954) with [7].

Table 1. Summary of experiments and impact coefficients ($a_v = 8.1$, $C_0 = 0.72$)

Experiment	Material of spheres and fluid	Solid density ρ_s (kg/m ³)	$\frac{\rho_s}{\rho_f}$	Fluid viscosity μ_f (NS/m ²)	Mean diameter spheres D (mm)	Solid volumetric concentration C	Sphere diameter to gap ratio $\lambda = \frac{D}{S}$	Impact coefficient a_i	
Bagnold (1954)	Wax and lead stearate mixture in water	1000	1.0	1×10^{-3}	1.32	0.135–0.555	1.3–11	0.009	
		1000	1.0	7×10^{-3}	1.32	0.555	11	0.009	
Savage & McKeown (1983)	Polystyrene in salt water	1029	1.0	1.1×10^{-3}	0.97	0.43	5.3	0.026	
						0.53	9.3	0.026	
					1.24	0.43	5.3	0.025	
						0.53	9.3	0.026	
						Ave.	0.026		
Savage & Sayed (1984)	Glass in air	2970	2400	1.8×10^{-5}	1.8	0.48	6.8	0.029	
						0.49	7.4	0.030	
						0.51	8.1	0.028	
							Ave.	0.029	
	Polystyrene in air	1095	900	1.8×10^{-5}	1.0	0.46	6.2	0.039	
						0.48	7.0	0.040	
							0.50	7.9	0.040
						1.32	0.44	5.7	0.043
						0.46	6.2	0.046	
						Ave.	0.042		

water. They subtracted the full plain fluid calibration stress τ_f from the total measured stress to yield particle impact and viscous shear stress. Bagnold subtracted a partial plain fluid stress $\tau_f/(1 + \lambda)$ by multiplying a factor of linear porosity, which is defined as $S/(S + D) = 1/(1 + \lambda)$, by τ_f . When $S = 0$ and $\lambda = D/S = \infty$, the mixture is a rigid structure and the shear stress in the interstitial fluid should vanish. Savage & McKeown (1983) mentioned that to their data $\tau_f - \tau_f/(1 + \lambda) = \tau_f \lambda/(1 + \lambda)$ should be added in order to compare with Bagnold's data. This adjustment has been made for comparison in this paper.

Savage & Sayed (1984) conducted shear tests with polystyrene and glass beads in air. When air is the interstitial fluid, due to the small dynamic viscosity μ_f of air compared to water, the plain fluid shear stress is negligible and the adjustment for fluid stress is no longer needed.

Table 1 summarizes these three sets of experiments and the impact coefficients a_i determined from the data. The viscous coefficient a_v is constant at 8.1. As shown in table 1, the impact coefficient a_i for the wax and lead stearate mixture in water is 0.009, for polystyrene in salt water it is 0.026, for glass in air it is 0.029 and for polystyrene in air it is 0.042. Using these a_i values in comparison of measured data and predictions by [7] is shown in figure 7.

For clarity, only $\lambda = 11$ of Bagnold's data is shown in figure 7. Bagnold's data for $\lambda = 1.3$ to 11 ($C = 0.135$ to 0.555) form a single line as predicted by [7], figure 6. These three sets of data in figure 7 cover a wide range of variables: concentration from 0.135 to 0.555, sphere diameter from 0.97 to 1.8 mm, solid density from 1000 to 2970 kg/m³ and viscosity from 1.8×10^{-5} to 7×10^{-3} NS/m². The solid materials were wax, polystyrene and glass beads. Equation [7] predicts the data well. Bagnold's equation has difficulties correlating data by Savage & McKeown (1983) and Savage & Sayed (1984).

The momentum exchange of a particle at impact is affected by the coefficient of restitution of the solid. Bagnold's sphere was made of wax and lead stearate mixture which has a low coefficient of restitution, hence a low $a_i = 0.009$. For a material having a high coefficient of restitution, such as Savage & McKeown's polystyrene beads in salt water, a_i increases to 0.026. The density and viscosity of the fluid also have influence. Spheres are easier to accelerate in low density air because the effect of added mass on the drag force is much smaller than in water. The motion of spheres should be livelier, momentum exchange and a_i should be larger in air than in heavier water. For Savage & Sayed's polystyrene beads in air, the impact coefficient is increased to $a_i = 0.042$ from 0.026 in water.

Savage & Sayed's (1984) tested glass beads in air. Due to many collisions, the surface of the brittle glass beads became pitted and roughened by many fractures. The fractured glass beads reduced

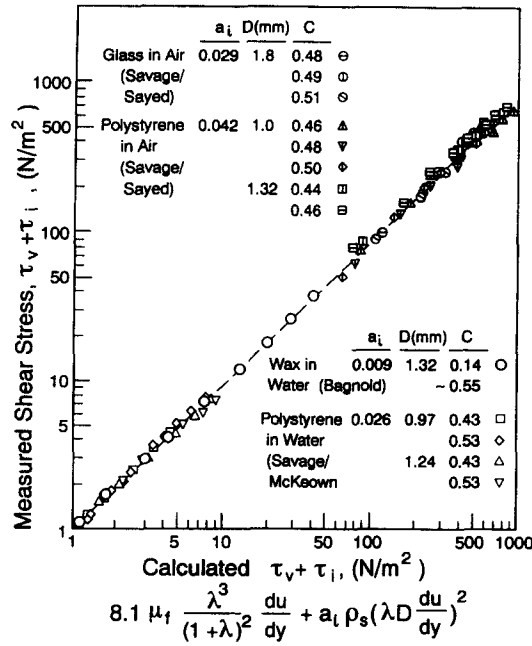


Figure 7. Comparison of several sets of measured shear stresses with [7].

the coefficient of restitution. This explains the decrease of a_i to 0.029 for glass beads in air when compared with smooth polystyrene beads in air at 0.042.

Even at the highest velocity gradient in Bagnold's measurements ($du/dy = 245 \text{ s}^{-1}$), the shearing motion is still not fully dominated by inertia with the power of $du/dy < 2$. The viscous shear stress is estimated about 30% of the measured stress. Bagnold (1954) indicated that $a_i = 0.0128$, which includes viscous shear. When the 30% viscous stress is subtracted, the impact coefficient is reduced to 0.009.

For Bagnold (1954) and Savage & McKeown's (1983) data on spheres in water, [7] is applicable up to $C = 0.555$. Above that the jamming of particles in the confined test apparatus causes the stress to increase higher than predicted in [7]. For spheres in air in a confined apparatus, the upper limiting concentration for [7] is about 0.53. The data which showed mechanical jamming with scattering is not included in figure 7.

4. TURBULENT SHEAR STRESS OF THE FLUID τ_f

At high concentrations, the turbulent shear stress in the interstitial fluid is usually small when compared with the sum of viscous and impact stresses. However, for low concentrations, both viscous and impact stresses are greatly reduced. The turbulence stress in the interstitial fluid becomes an important term.

The turbulent shear stress in the interstitial fluid can be represented by the Reynold's stress $-\rho_f \overline{u'v'}$, in which u' and v' are turbulent velocity components. The amount of shearing action of the fluid in the mixture is reduced by the presence of a solid phase. A reasonable reduction factor is the linear porosity $S/(S + D) = 1/(1 + \lambda)$ used by Bagnold. The turbulent shear stress in the fluid can be expressed as

$$\tau_f = -\frac{\rho_f \overline{u'v'}}{(1 + \lambda)}. \tag{10}$$

Based on measurements in solid-fluid mixtures in conduits, the velocity profile is still proportional to the logarithm of the distance from the boundary, the same trend as for the plain fluid (Daily & Hardison 1964; Roberts *et al.* 1967; Mih & Parker 1967). This indicates that the Karman-Prandtl approach for turbulent fluid is applicable in the mixture. Therefore,

$$|u'| \sim |-v'| \sim l \frac{du}{dy} = ky \frac{du}{dy}, \tag{11}$$

where l is the mixing length, $k = 0.4$ is Karman's constant for pure fluid, y is the distance away from the wall and du/dy is the velocity gradient at a point. Substituting [11] into [10]:

$$\tau_f = \rho_f \left(0.4y \frac{du}{dy} \right)^2 \frac{1}{1 + \lambda} \quad [12]$$

Equation [12] is for conduit and channel flows. For a cylindrical Couette flow, the shear is nearly constant across the gap of the Couette apparatus, [12] can be rewritten as

$$\tau_f = a_f \rho_f \left(h \frac{du}{dy} \right)^2 \frac{1}{1 + \lambda}, \quad [13]$$

in which a_f is the fluid coefficient and h is the half gap; the plane of symmetry for the velocity profile in a Couette flow is at mid-gap. Based on plain fluid Couette tests by Bagnold (1954) and Savage & McKeown (1983), the average fluid coefficient is $a_f = 0.0025$. Adding [7] and [13], the equation for shear stress in a non-cohesive solid-fluid mixture Couette flow is

$$\tau = 8.1\mu_f \frac{\lambda^3}{(1 + \lambda)^2} \frac{du}{dy} + a_i \rho_s \left(\lambda D \frac{du}{dy} \right)^2 + 0.0025 \rho_f \left(h \frac{du}{dy} \right)^2 \frac{1}{1 + \lambda}, \quad [14]$$

where a_i is a function of solid materials and fluids as listed in table 1. For conduit and channel flows, the last term of [14], τ_f , should be replaced by [12].

5. CONCLUSION

This analysis provides a new equation, [7], for the viscous interaction shear stress in solid-fluid mixture flow which is supported by several sets of experiments covering a wide range of physical variables. Previous analyses do not accurately correlate with the experiments. Together with an equation for turbulence in the interstitial fluid, [14] is a new shear equation for non-cohesive mixture flow.

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